Bibliography

This bibliograpy is an extract from “Grassmann Algebra Volume 1: Foundations - Exploring extended vector algebra with *Mathematica*” by John Browne. First Edition 2012.

The scope of this bibliography

The scope of this bibliography is limited. Grassmann algebra has just too many references in recent times, particularly since the expansion of the web, to entertain any sort of completeness. Nevertheless, up until the end of the twentieth century there were still only a few dozen important references *in English* to the *foundational* aspects of the algebra. Since this book (Volume 1) is limited to discussing these foundations, the bibliography will be limited accordingly: to relevant publications appearing before the year 2000. Other applications of the algebra, to calculus and mechanics, and to Clifford and hypercomplex algebras will be discussed in later volumes. This bibliography only covers these other applications sparsely to provide context on the related research activity during the period.

Since the advent of the web, it is now possible to find much by a simple search. On the other hand a simple search does not return what it has not found. And this becomes more critical for older references. The bibliography therefore lays more emphasis on earlier works.

Although mechanics is covered later rather than in this volume, the bibliography also includes some obscure works on mechanics using line geometry, screws and motor algebra. These are included here because the results may also be obtained using the bound vectors and bivectors discussed in this volume.

Armstrong H L 1959

‘On an alternative definition of the vector product in *n*-dimensional vector analysis’
Matrix and Tensor Quarterly, **IX** no 4, pp 107-110.

The author proposes a definition equivalent to the complement of the exterior product of $n-1$ vectors.

Ball R S 1876

The Theory of Screws: A Study in the Dynamics of a Rigid Body
Dublin

See the note on ‘The Directional Theory of Screws’ [Hyde 1888].

Ball R S 1900

A Treatise on the Theory of Screws
Cambridge University Press (1900). Reprinted 1998.

A classical work on the theory of screws, containing an annotated bibliography in which Ball refers to the *Ausdehnungslehre* of 1862: “This remarkable work … contains much that is of instruction and interest in connection with the present theory … . Here we have a very general theory which includes screw coordinates as a special case.” Ball does not use Grassmann’s methods in his treatise.

Barnabei M, Brini A and Rota G-C 1985

‘On the Exterior Calculus of Invariant Theory’
Journal of Algebra, **96**, pp 120-160.

Contents are: Introduction, Peano spaces, The join, The meet, Cap-products, The Hodge star operator, Alternative laws, Geometric calculus.

Barton H 1927

'A Modern Presentation of Grassmann's Tensor Analysis'
American Journal of Mathematics, **XLIX**, pp 598-614.

This paper covers similar ground to that of Moore (1926).

Birss R R 1980

'Multivector Analysis I: A Comparison with Tensor Algebra'
Physics Letters, **78A**, No 3, pp 223-226.

'Multivector Analysis II: A Comparison with Vector Algebra'
Physics Letters, **78A**, No 3, pp 227-230.

The comparisons are with reference to crystal physics.

Bourbaki N 1948

Algèbra
Actualités Scientifiques et Industrielles No.1044, Paris

Chapter III treats multilinear algebra.

Bowen R M and Wang C-C 1976

Introduction to Vectors and Tensors
Plenum Press. Two volumes.

This is one of the few contemporary texts on vectors and tensors which relates points and vectors via the explicit introduction of the origin into the calculus (p 254).

Brand L 1947

Vector and Tensor Analysis
Wiley, New York.

Contains a chapter on motor algebra which, according to the author in his preface "… is apparently destined to play an important role in mechanics as well as in line geometry". There is also a chapter on quaternions.

Buchheim A 1884-1886

'On the Theory of Screws in Elliptic Space'
Proceedings of the London Mathematical Society
xiv (1884) pp 83-98, **xvi** (1885) pp 15-27, **xvii** (1886) pp 240-254, **xvii** p 88.

The author writes "My special object is to show that the *Ausdehnungslehre* supplies all the necessary materials for a calculus of screws in elliptic space. Clifford was apparently led to construct his theory of biquaternions by the want of such a calculus, but Grassmann's method seems to afford a simpler and more natural means of expression than biquaternions." (*xiv*, p 90) Later he extends this theory to "… all kinds of space." (*xvi*, p 15)

Burali-Forti C 1897

Introduction à la Géométrie Différentielle suivant la Méthode de H. Grassmann
Gauthier-Villars, Paris

This work covers both algebra and differential geometry in the tradition of the Peano approach to the *Ausdehnungslehre*.

Burali-Forti C and Marcolongo R 1910

Éléments de Calcul Vectoriel
Hermann, Paris

Mostly a treatise on standard vector analysis, but it does contain an appendix (pp 176-198) on the methods of the *Ausdehnungslehre* and some interesting historical notes on the vector calculi and their notations. The authors use the wedge $\wedge$ to denote Gibbs' cross product $\times$ and use the $\times$, initially introduced by Grassmann to denote the scalar or inner product.

Available as a print-on-demand volume.

Cartan É 1922

Leçons sur les Invariants Intégraux
Hermann, Paris

In this work Cartan develops the theory of exterior differential forms, remarking that he has called them '*extérieures'* since they obey Grassmann's rules of '*multiplication extérieures'*.

Cartan É 1938

Leçons sur la Théorie des Spineures
Hermann, Paris

In the introduction Cartan writes "One of the principal aims of this work is to develop the theory of spinors systematically by giving a purely geometrical definition of these mathematical entities: because of this geometrical origin, the matrices used by physicists in quantum mechanics appear of their own accord, and we can grasp the profound origin of the property, possessed by Clifford algebras, of representing rotations in space having any number of dimensions". Contains a short section on multivectors and complements (the French term is *supplément*).

An English translation was published by Hermann of Paris in 1966 as *The Theory of Spinors*, and republished in 1981 by Dover.

Cartan É 1946

Leçons sur la Géométrie des Espaces de Riemann
Gautier-Villars, Paris

This is a classic text (originating in 1925) on the application of the exterior calculus to differential geometry. Cartan begins with a chapter on exterior algebra and its expression in tensor index notation. He uses square brackets to denote the exterior product (as did Grassmann), and a wedge to denote the cross product.

Carvallo M E 1892

'La Méthode de Grassmann'
Nouvelles Annales de Mathématiques, serie 3 **XI**, pp 8-37.

An exposition of some of Grassmann's methods applied to three dimensional geometry following the approach of Peano. It does not treat the interior product.

Chevalley C 1955

The Construction and Study of Certain Important Algebras
Mathematical Society of Japan, Tokyo.

Lectures given at the University of Tokyo on graded, tensor, Clifford and exterior algebras.

Clifford W K 1873

'Preliminary Sketch of Biquaternions'
Proceedings of the London Mathematical Society, **IV**, nos 64 and 65, pp 381-395.

This paper includes an interesting discussion of the geometric nature of mechanical quantities. Clifford adopts the term 'rotor' for the bound vector, and 'motor' for the general sum of rotors. By analogy with the quaternion as a quotient of vectors he defines the biquaternion as a quotient of motors.

It is in this paper also that Clifford introduces the symbol ω with the *formal* property that $\omega^2 = 0$, thus enabling a whole new spectrum of quantities to be defined for representing physical phenomena.

Clifford W K 1878

'Applications of Grassmann's Extensive Algebra'
American Journal of Mathematics Pure and Applied, **I**, pp 350-358.

In this paper Clifford lays the foundations for general Clifford algebras.

Clifford W K 1882

Mathematical Papers
Reprinted by Chelsea Publishing Co, New York (1968).

Of particular interest in addition to his two published papers above are the otherwise unpublished notes:
'Notes on Biquaternions' (~1873)
'Further Note on Biquaternions' (1876)
'On the Classification of Geometric Algebras' (1876)
'On the Theory of Screws in a Space of Constant Positive Curvature' (1876).

Coffin J G 1909

Vector Analysis
Wiley, New York.

This is the second English text in the Gibbs-Heaviside tradition. It contains an appendix comparing the various notations in use at the time, including his view of the Grassmannian notation.

Collins J V 1899-1900

'An elementary Exposition of Grassmann's *Ausdehnungslehre* or Theory of Extension'
American Mathematical Monthly,
6 (1899) pp 193-198, 261-266, 297-301; **7** (1900) pp 31-35, 163-166, 181-187, 207-214, 253-258.

This work follows in summary form the *Ausdehnungslehre* of 1862 as regards general theory but differs in its discussion of applications. It includes applications to geometry and brief applications to linear equations, mechanics and logic.

These have been collected into a slim print-on-demand volume.

Coolidge J L 1940

'Grassmann's Calculus of Extension' in
A History of Geometrical Methods
Oxford University Press, pp 252-257.

This brief treatment of Grassmann's work is characterized by its lack of clarity. The author variously describes an exterior product as "essentially a matrix" and as "a vector perpendicular to the factors" (p 254). And confusion arises between Grassmann's matrix and the division of two exterior products (p 256).

Cox H 1882

'On the Application of Quaternions and Grassmann's *Ausdehnungslehre* to different kinds of Uniform Space'
Cambridge Philosophical Transactions, **XIII** part II, pp 69-143.

The author shows that the exterior product is the multiplication required to describe non-metric geometry, for "it involves no ideas of distance" (p 115). He then discusses exterior, regressive and interior products, applying them to geometry, systems of forces, and linear complexes – using the notation of 1844. In other papers Cox applies the *Ausdehnungslehre* to non-Euclidean geometry (1873) and to the properties of circles (1890).

Crowe M J 1967, 1985

A History of Vector Analysis
Notre Dame 1967. Republished by Dover 1985.

This is the most informative work available on the history of vector analysis from the discovery of the geometric representation of complex numbers to the development of the Gibbs-Heaviside system. Crowe's thesis is that the Gibbs-Heaviside system grew mostly out of quaternions rather than from the *Ausdehnungslehre*. His explanation of Grassmannian concepts is particularly accurate in contradistinction to many who supply a more casual reference.

Dibag I 1974

'Factorization in Exterior Algebras'
Journal of Algebra, **30**, pp 259-262

The author develops necessary and sufficient conditions for an *m*-element to have a certain number of 1-element factors. He also shows that an (*n*–2)-element in an odd dimensional space always has a 1-element factor.

Dimentberg F M 1965

The Screw Calculus and its Applications in Mechanics
Translated by the Foreign Technology Division of the National Technical Information Service, U.S. Department of Commerce. Document number: FTD-HT-23-1632-67.

Dimentberg uses Clifford's dual number approach to develop screw theory applied to mechanics.

Drew T B 1961

Handbook of Vector and Polyadic Analysis
Reinhold, New York

Tensor analysis in invariant notation. Of particular interest here is a section (p 57) on 'polycross products' – an extension of the (three-dimensional) cross product to polyads.

Efimov N V and Rozendorn E R 1975

Linear Algebra and Multi-Dimensional Geometry
MIR, Moscow

Contains a chapter on multivectors and exterior forms.

Fehr H 1899

Application de la Méthode Vectorielle de Grassmann à la Géométrie Infinitésimale
Georges Carré, Paris

Thesis comprising an initial chapter on exterior algebra as well as standard differential geometry. Available as a print-on-demand volume.

Fleming W H 1965

Functions of Several Variables
Addison-Wesley

Contains a chapter on exterior algebra.

Forder H G 1941

The Calculus of Extension
Cambridge (also reprinted by Chelsea)

This text is one of the most recent of the few devoted to an exposition of Grassmann's methods. It is an extensive work (490 pages) largely using Grassmann's notations and relying primarily on the *Ausdehnungslehre* of 1862. Its application is particularly to geometry including many examples well illustrating the power of the methods, a chapter on forces, screws and linear complexes, and a treatment of the algebra of circles.

Gibbs J W 1886

'On multiple algebra'
Address to the American Association for the Advancement of Science
In *Collected Works*, Gibbs 1928, vol 2.

This paper is probably the most authoritative historical comparison of the different 'vectorial' algebras of the time. Gibbs was obviously very enthusiastic about the *Ausdehnungslehre*, and shows himself here to be one of Grassmann's greatest proponents.

Gibbs J W 1891

'Quaternions and the Ausdehnungslehre'
Nature, **44**, pp 79–82. Also in *Collected Works*, Gibbs 1928.

Gibbs compares Hamilton's Quaternions with Grassmann's *Ausdehnungslehre* and concludes that "... Grassmann's system is of indefinitely greater extension ...". Here he also concludes that to Grassmann must be attributed the discovery of matrices. Gibbs published a further three papers in *Nature* (also in *Collected Works*, Gibbs 1928) on the relationship between quaternions and vector analysis, providing an enlightening insight into the quaternion–vector analysis controversy of the time.

Gibbs J W 1928

The Collected Works of J. Willard Gibbs Ph.D. LL.D.
Two volumes. Longmans, New York.

In part 2 of Volume 2 is reprinted Gibbs' only personal work on vector analysis: *Elements of Vector Analysis, Arranged for the Use of Students of Physics* (1881–1884). This was not published elsewhere.

Grassmann H G 1844

Die lineale Ausdehnungslehre: ein neuer Zweig der Mathematik
Leipzig.

The full title is *Die lineale Ausdehnungslehre: ein neuer Zweig der Mathematik dargestellt und durch Andwendungen auf die übrigen Zweige der Mathematik, wie auch auf die Statik, Mechanik, die Lehre vom Magnetismus und die Krystallonomie erläutert*. This first book on Grassmann's new mathematics is known shortly as *Die Ausdehnungslehre von 1844*. It develops the theory of exterior multiplication and division and regressive exterior multiplication. It does not treat complements or interior products in the way of the *Ausdehnungslehre* of 1862. The original *Die Ausdehnungslehre von 1844* was republished in 1878. The best source to this work is Volume 1 of Grassmann's collected works (1896), of which an English translation has been made by Lloyd C. Kannenberg (1995).

Grassmann H G 1845

'Kurze Übersicht über das Wesen der Ausdehnungslehre'
Archiv der Mathematik und Physik (Grunert's Archiv) **VI**

This is a review written by Grassmann, requested by J A Grunert, of his new book published in 1844.

The review was translated by W W Beman and published as 'A Brief Account of the Essential Features of Grassmann's Extensive Algebra' in the The Analyst, VIII, 1881, pp 96-97, 114-124.

Grassmann H G 1855

'Sur les différents genres de multiplication'
Crelle's Journal, **49**, pp 123-141.

This paper was written to claim of Cauchy priority for a method of solving linear equations.

Grassmann H G 1862

Die Ausdehnungslehre. Vollständig und in strenger Form
Berlin.

This is Grassmann's second attempt to publish his new discoveries in book form. It adopts a substantially different approach to the *Ausdehnungslehre* of 1844, relying more on the theorem–proof approach. The work comprises two main parts: the first on the exterior algebra and the second on the theory of functions. The first part includes chapters on addition and subtraction, products in general, combinatorial products (exterior and regressive), the interior product, and applications to geometry. This is probably Grassmann's most important work. The best source is Volume 1 of Grassmann's collected works (1896), of which an English translation has been made by Lloyd C. Kannenberg (2000). In the collected works edition, the editor Friedrich Engel has appended extensive notes and comments, which Kannenberg has also translated.

Short excerpts were translated by M Kormes and published in *A Source Book in Mathematics* (ed D E Smith), McGraw-Hill, 1929, pp 684-696.

Grassmann H G 1878

'Verwendung der Ausdehnungslehre für die allgemeine Theorie der Polaren und den Zusammenhang algebraischer Gebilde'
Crelle's Journal, **84**, pp 273–283.

This is Grassmann's last paper. It contains, among other material, his most complete discussion on the notion of 'simplicity'.

Grassmann H G 1896–1911

Hermann Grassmanns Gesammelte Mathematische und Physikalische Werke
Teubner, Leipzig. Volume 1 (1896), Volume 2 (1902, 1904), Volume 3 (1911).

Grassmann's complete collected works appeared between 1896 and 1911 under the editorship of Friedrich Engel and with the collaboration of Jakob Lüroth, Eduard Study, Justus Grassmann, Hermann Grassmann jr. and Georg Scheffers. The following is a summary of their contents.

Volume 1
Die lineale Ausdehnungslehre: ein neuer Zweig der Mathematik (1844)
Geometrische Analyse: geknüpft an die von Leibniz erfundene geometrische Charakteristik (1847)
Die Ausdehnungslehre. Vollständig und in strenger Form (1862)

Volume 2
Papers on geometry, analysis, mechanics and physics

Volume 3
Theorie der Ebbe und Flut (1840)
Further papers on mathematical physics.

Parts of Volume 1 (*Die lineale Ausdehnungslehre* (1844) and *Geometrische Analyse* (1847)) together with selected papers on mathematics and physics have been translated into English by Lloyd C Kannenberg and published as *A New Branch of Mathematics* (1995). *Geometrische Analyse* is Grassmann's prize-winning essay fulfilling Leibniz' search for an algebra of geometry. The remainder of Volume 1 (*Die Ausdehnungslehre* (1862)) has been translated into English by Lloyd C Kannenberg and published as *Extension Theory* (2000).

Volume 2 comprises papers on geometry, analysis, analytical mechanics and mathematical physics plus two texts, one on arithmetic and the other on trigonometry. Available as a print-on-demand volume.

Volume 3 comprises Grassmann's earliest major work (*Theorie der Ebbe und Flut* (1840)) and further papers, particularly on wave theory. The Theory of Tides begins to apply Grassmann's new approach to vector analysis.

Grassmann H der Jüngere 1909, 1913, 1927

Projektive Geometrie der Ebene
Teubner, Leipzig.

A comprehensive treatment in three books using the methods of the elder H. Grassmann.

Grassmann R 1895

Die Ausdehnungslehre oder die Wissenschaft von den extensiven Grösen in strenger Formelentwicklung
Stettin.

This is one of the volumes in a series by Hermann Grassmann's younger brother Robert, who published over 40 books on diverse subjects. See the paper by Ivor Grattan-Guinness in Petsche (2011) for an analysis of Robert's work.

Greenberg M J 1976

'Element of area via exterior algebra'
American Mathematical Monthly, **83**, pp 274-275.

The author suggests that the treatment of elements of area by using the exterior product would be a more satisfactory treatment than that normally given in calculus texts.

Greub W H 1967

Multilinear Algebra
Springer-Verlag, Berlin.

Contains chapters on exterior algebra.

Gurevich G B 1964

Foundations of the Theory of Algebraic Invariants
Noordhoff, Groningen

Contains a chapter on *m*-vectors (called here polyvectors) with an extensive treatment of the conditions for divisibility of an *m*-vector by one or more vectors (pp 354-395).

Hamilton W R 1853

Lectures on Quaternions
Dublin

The first English text on quaternions. The introduction briefly mentions Grassmann.

Hamilton W R 1866

Elements of Quaternions
Dublin

The editor Charles Joly in his preface remarks in relation to the quaternion's associativity that "For example, Grassmann's multiplication is sometimes associative, but sometimes it is not". The exterior product is of course associative. However, here it seems Joly may be suffering from a confusion caused by Grassmann's notation for expressions in which both exterior and regressive products appear.
The second edition of 1899 has been reprinted in 1969 by Chelsea Publishing Company.

Hardy A S 1895

Elements of Quaternions
Ginn and Company, Boston

The author claims this to be an introduction to quaternions at an elementary level.

Heath A E 1917

'Hermann Grassmann 1809-1877'
The Monist, **27**, pp 1-21.

'The Neglect of the Work of H. Grassmann'
The Monist, **27**, pp 22-35.

'The Geometric Analysis of Hermann Grassmann and its connection with Leibniz's characteristic'
The Monist, **27**, pp 36-56.

Hestenes D 1966

Space–Time Algebra
Gordon and Breach

This work is a seminal exposition of Clifford algebra emphasizing the geometric nature of the quantities and operations involved. The author writes "… ideas of Grassmann are used to motivate the construction of Clifford algebra and to provide a geometric interpretation of Clifford numbers. This is to be contrasted with other treatments of Clifford algebra which are for the most part formal algebra. By insisting on Grassmann's geometric viewpoint, we are led to look upon the Dirac algebra with new eyes." (p 2).

Hestenes D 1968

'Multivector Calculus'
Journal of Mathematical Analysis and Applications, **24**, pp 313-325.

In the words of the author: "The object of this paper is to show how differential and integral calculus in many dimensions can be greatly simplified using Clifford algebra."

Hestenes D 1999

New Foundations for Classical Mechanics (Second Edition)
Kluwer

In the words of the author: "This is a textbook on classical mechanics at the intermediate level, but its main purpose is to serve as an introduction to a new mathematical language for physics called *geometric algebra*." Geometric algebra and Grassmann algebra are intimately related via the Clifford product.

Hodge W V D 1952

Theory and Applications of Harmonic Integrals
Cambridge

The 'star' operator defined by Hodge has a precursor of similar nature in Grassmann's complement operator.

Hunt K H 1970

Screw systems in Spatial Kinematics (Screw Systems Surveyed and Applied to Jointed Rigid Bodies)
Report MMERS 3, Department of Mechanical Engineering, Monash University, Clayton, Australia

Although in this report the author has intentionally confined himself to well known methods of pure and analytical geometry, he is a strong proponent of the screw as being the natural language for investigating spacial mechanisms.

Hyde E W 1884

‘Calculus of Direction and Position’
American Journal of Mathematics, **VI**, pp 1-13.

In this paper the author compares quaternions to the methods of the *Ausdehnungslehre* and concludes that Grassmann’s system is far preferable as a system of directed quantities.

Hyde E W 1888

‘Geometric Division of Non-congruent Quantities’
Annals of Mathematics, **4**, pp 9-18.

This paper deals with the concept of exterior division more extensively than did Grassmann.

Hyde E W 1888

‘The Directional Theory of Screws’
Annals of Mathematics, **4**, pp 137-155.

This paper is an account of the theory of screws using the *Ausdehnungslehre*. Hyde claims that “A screw evidently belongs thoroughly to the realm of the Directional Calculus and will not be easily or naturally treated by Cartesian methods; and Ball’s treatment is throughout essentially Cartesian in its nature.” Here he is referring to *The Theory of Screws: A Study in the Dynamics of a Rigid Body* (1876). In *A Treatise on the Theory of Screws* (1900) Ball comments: “Prof. Hyde proves by his [sic] calculus many of the fundamental theorems in the present theory in a very concise manner.” (p 531).

Hyde E W 1890

The Directional Calculus based upon the Methods of Hermann Grassmann
Ginn and Company, Boston

The author discusses geometric applications in 2 and 3 dimensions including screws and complements of bound elements (for example, points, lines and planes).

Hyde E W 1906

Grassmann’s Space Analysis
Wiley, New York

In the words of the author: “This little book is an attempt to present simply and concisely the principles of the ‘Extensive Analysis’ as fully developed in the comprehensive treatises of Hermann Grassmann, restricting the treatment however to the geometry of two and three dimensional space.”

Jahnke E 1905

Vorlesungen über die Vektorenrechnung mit Anwendungen auf Geometrie, Mechanik und Mathematische Physik.
Teubner, Leipzig

This work is full of examples of application of the *Ausdehnungslehre* (notation of 1862) to geometry, mechanics and physics. Only two and three dimensional problems are considered.

Kálnay A J 1976

'A Note on Grassmann Algebras'
Reports on Mathematical Physics, **9**, pp 9-13.

A report on two applications to physics.

Klein F 1908

Elementary Mathematics from an Advanced Standpoint (Volume 2) Geometry
Translated from the third German edition by E R Hedrick and C A Noble.
Dover 1939

Klein begins his discussion of geometric manifolds using Grassmannian concepts. But since he found the style of Grassmann's earlier work (1844) "extraordinarily obscure", he introduces the reader to them using determinant notation.

Lasker E 1896

'An Essay on the Geometrical Calculus'
Proceedings of the London Mathematical Society, **XXVIII**, pp 217-260.

This work differs from most of the other papers of this era on the geometrical applications of the *Ausdehnungslehre* by concentrating on a space of arbitrary dimension rather than two or three.

Lewis G N 1910

'On four-dimensional vector calculus and its application in electrical theory'
Proceedings of the American Academy of Arts and Sciences, **XLVI**, pp 165-181.

A specialization of the *Ausdehnungslehre* to four dimensions and its applications to electromagnetism in Minkowskian terms. The author introduces the new concepts with the minimum of explanation: for example, the anti-symmetric properties of bivectors and trivectors are justified as conventions! (p 167).

Lotze A 1922

Die Grundgleichungen der Mechanik insbesondere Starrer Körper
Teubner, Leipzig

This short monograph is one of the rare works addressing mechanics using the methods of the *Ausdehnungslehre*. It treats the dynamics of systems of particles and the kinematics and dynamics of rigid bodies.

Lounesto P 1997

Clifford Algebras and Spinors
Cambridge

Contains a preparatory chapter on bivectors and exterior algebra.

Macfarlane A 1904

Bibliography of Quaternions and Allied Systems of Mathematics
Dublin

Published for the *International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics*, this bibliography together with supplements to 1913 contains about 2500 articles including many on the *Ausdehnungslehre* and vector analysis.

Marcus M 1966

'The Cauchy-Schwarz inequality in the exterior algebra'
Quarterly Journal of Mathematics, **17**, pp 61-63.

The author shows that a classical inequality for positive definite hermitian matrices is a special case of the Cauchy-Schwarz inequality in the appropriate exterior algebra.

Marcus M 1975

Finite Dimensional Multilinear Algebra
Marcel Dekker, New York

Part II contains chapters on Grassmann and Clifford algebras.

Marcus M and Robinson H 1975

'A Note on the Hodge Star Operator'
Linear Algebra and its Applications, **10**, pp 85-87.

In the author's words: "The usual proof that the Hodge star operator on the Grassmann algebra is independent of the orthonormal basis used to define it requires a decomposition of the operator into a product of three maps, each of which is independent of the basis. The present note contains a very short proof of the result which depends on simple properties of the multiplication and induced inner product in the Grassmann algebra."

Marcus M 1978

'An Inequality for Non-decomposable Elements in the Grassmann Algebra'
Houston Journal of Mathematics, **4**, No 3, pp 417-422.

Here the term 'decomposable' is equivalent to the term 'simple' used in this book. (See also Soule, 1979)

Massey W S 1983

'Cross Products of Vectors in Higher Dimensional Euclidean Spaces'
The American Mathematical Monthly, **90**, No 10, pp 697-701.

The author shows that "… a cross product of vectors exists only in 3-dimensional and 7-dimensional space."

Mehmke R 1880

Andwendung Der Grassmann'schen Ausdehnungslehre Auf Die Geometrie Der Kreise In Ebene
Stuttgart

This is probably the first application of Grassmannian methods to the geometry of circles.
Delivered as his inaugural dissertation at the University of Tübingen. Available as a print-on-demand publication.

Mehmke R 1913

Vorlesungen über Punkt- und Vektorenrechnung (2 volumes)
Teubner, Leipzig

Volume 1 (394 pages) deals with the analysis of bound elements (points, lines and planes) and projective geometry.

Milne E A 1948

Vectorial Mechanics
Methuen, London

An exposition of three-dimensional vectorial (and tensorial) mechanics using 'invariant' notation. Milne is one of the rare authors who realizes that physical forces and the linear momenta of particles are better modelled by line vectors (bound vectors) rather than by the usual vector algebra. He treats systems of line vectors (bound vectors) as vector pairs.

Moore C L E 1926

'Grassmannian Geometry in Riemannian Space'
Journal of Mathematics and Physics, **5**, pp 191-200.

This paper treats the complement, and the exterior, regressive and interior products in a Riemannian space in tensor index notation using the alternating tensors and the generalized Kronecker symbol. This classic use of tensor notation does not enhance the readability of the exposition.

Murnaghan F D 1925

'The Generalised Kronecker Symbol and its Application to the Theory of Determinants'
American Mathematical Monthly, **32**, pp 233-241.

The generalized Kronecker symbol is essentially the generalization to an exterior product space of the usual Kronecker symbol.

Murnaghan F D 1925

'The Tensor Character of the Generalised Kronecker Symbol'
Bulletin of the American Mathematical Society, **31**, pp 323-329.

The author states "It will be readily recognized that there is an intimate connection here with Grassmann's *Ausdehnungslehre*, and we believe, in fact, that a systematic exposition of this theory with the aid of the generalized Kronecker symbol would help to make it more widely understood."

Park D 2010

'A precedence operator for the *Mathematica* implementation of *GrassmannAlgebra*'
'Documentation of the *GrassmannAlgebra* application using *WolframWorkbench*'
Private communications.

The results of this work have been incorporated into the *GrassmannAlgebra* application.
David Park has provided a number of resources for the *Mathematica* community.
They may be found at http://home.comcast.net/~djmpark/Mathematica.html

Peano G 1888

Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann
Fratelli Bocca, Torino

This work has been translated into English by Lloyd C Kannenberg and published in 2000 by Birkhauser under the title *Geometric Calculus According to the Ausdehnungslehre of H, Grassmann*.

Peano G 1895

‘Essay on Geometrical Calculus’
in *Selected Works of Guiseppe Peano* Chapter XV, p 169-188.
Allen and Unwin, London

The essay is translated from ‘Saggio di calcolo geometrico’ *Atti, Accad. Sci. Torino*, **31**, (1895-6) pp 952-975. Peano claims to have understood Grassmann’s ideas by reconstructing them himself. His geometric ideas come through with clarity, substantiating his claim. Peano’s principle exposition of Grassmann’s work was in *Calcolo geometrico secondo l’Ausdehnunglehre di H. Grassmann*, Turin (1888) of which a translation of selected passages appears on pp 90-100 of the above *Selected Works*.

Peano G 1901

Formulaire de Mathématiques
Gauthier-Villars, Paris

A compendium of axioms and results. The last part (pp 192-209) is devoted to point and vector spaces and includes some interesting historical comments.

Pedoe D 1967

‘On a geometrical theorem in exterior algebra’
Canadian Journal of Mathematics, **19**, pp 1187-1191.

The author remarks “This paper owes its inspiration to the remarkable book by H. G. Forder *The Calculus of Extension* … Forder introduces many concepts which I find difficult to bring down to earth. But the methods developed in his book are powerful ones, and it is evident that much work can usefully be done in simplifying and interpreting some of the concepts he uses.” He does not mention Grassmann.

Petsche H-J 2009

Hermann Graßmann Biography
Birkhäuser, Berlin

Translated by Mark Minnes. The first volume of a three volume set commemorating the 200th anniversary of Grassmann’s birth in 1809.

Petsche H-J, Kanneberg L, Keßler G, and Liskowacka J (eds) 2009

Hermann Graßmann Roots and Traces
Birkhäuser, Berlin

Subtitled: “Autographs and Unknown Documents”. The second volume of a three volume set commemorating the 200th anniversary of Grassmann’s birth in 1809.

Petsche H-J, Lewis A C, Liesen J, and Russ S (eds) 2011

Hermann Graßmann From Past to Future: Graßmann’s Work in Context
Graßmann Bicentennial Conference, September 2009
Birkhäuser, Berlin

The third volume of a three volume set commemorating the 200th anniversary of Grassmann’s birth in 1809. These are the papers from the conference.

Saddler W 1927

‘Apolar triads on a cubic curve’
Proceedings of the London Mathematical Society, Series 2, **26**, pp 249-256.

‘Apolar tetrads on the Grassmann quartic surface’
Journal of the London Mathematical Society, **2**, pp 185-189.

Exterior algebra applied to geometric construction.

Sain M 1976

‘The Growing Algebraic Presence in Systems Engineering: An Introduction’
Proceedings of the IEEE, **64**, p 96.

A modern algebraic discussion culminating in the definition of the exterior algebra and its relationship to the theory of determinants and some systems theoretical applications.

Schlegel V 1872, 1875

System der Raumlehre (2 volumes)
Leipzig

Geometry using Grassmann’s methods. Available as print-on-demand volumes.

Schouten J A 1951

Tensor Analysis for Physicists
Oxford

Contains a chapter on *m*-vectors from a tensor-analytic viewpoint.

Schubring G (ed) 1996

Hermann Günther Graßmann (1809-1877): Visionary Mathematician, Scientist and Neohumanist Scholar
Kluwer

Papers from a sesquicentennial conference on 150 years of Grassmann’s work, held in 1994 in Lieschow near his birthplace Stettin (Szczecin).

Schweitzer A R 1950

Bulletin of the American Mathematical Society, **56**
‘Grassmann’s extensive algebra and modern number theory’ (Part I, p 355; Part II, p 458)
‘On the place of the algebraic equation in Grassmann’s extensive algebra’ (p 459)
‘On the derivation of the regressive product in Grassmann’s geometrical calculus’ (p 463)
‘A metric generalisation of Grassmann’s geometric calculus’ (p 464)

Resumés only of these papers are printed.

Scott R F 1880

A Treatise on the Theory of Determinants
Cambridge, London

The author states in the preface that “The principal novelty in the treatise lies in its systematic use of Grassmann’s alternate units, by means of which the study of determinants is, I believe, much simplified.”

Shepard G C 1966

Vector Spaces of Finite Dimension
Oliver and Boyd, London

Chapter IV contains an introduction to exterior products via tensors and multilinear algebra.

Soule G W 1979

'An Inequality in the Grassmann Algebra'
Houston Journal of Mathematics, **5**, No 2, pp 269-275.

Here the term 'decomposable' is equivalent to the term 'simple' used in this book. (See also Marcus, 1978)

Thomas J M 1962

Systems and Roots
W Byrd Press

The author uses exterior algebraic concepts in some of his network analysis.

Tonti E 1972

Accademia Nazionale die Lincei, Serie VIII, Volume L II
'On the Mathematical Structure of a Large Class of Physical Theories' (Fasc.1, p 48)
'A Mathematical Model for Physical Theories' (Fasc.2-3, p 176, 351)

These papers begin the author's investigations into the structure of physical theories, in which he considers the geometrical calculus to play an important part.

Tonti E 1975

On the Formal Structure of Physical Theories
Report of the *Instituto di Matematica del Politecnico di Milano*

In this report the author constructs a classification scheme for physical quantities and the equations of physical theories. The mathematical structures needed for this, and which are reviewed in this report are algebraic topology, exterior algebra, exterior differential forms, and Clifford algebra. The author shows that the underlying structure of physical theories is basically capable of a geometric interpretation.

von Mises R 1924

Motor Calculus A new theoretical device for mechanics
Translated by E J Baker and K Wohlhart 1996
Institute for Mechanics, University of Technology Graz

A translation of two long articles. The term 'motor' was introduced by Clifford to designate a screw with an associated magnitude. But von Mises eschews the use of Clifford's duality unit and instead uses the concept of ordered line pairs. He does not use Grassmannian methods.

Whitehead A N 1898

A Treatise on Universal Algebra (Volume 1)
Cambridge

No further volumes appeared. This is probably the best and most complete exposition of Grassmann's works in English (586 pages). The author recreates many of Grassmann's results, in many cases extending and clarifying them with original contributions. Whitehead considers non-Euclidean metrics and spaces of arbitrary dimension. However, like Grassmann, he does not distinguish between *n*-elements and scalars.

Willmore T J 1959

Introduction to Differential Geometry
Oxford

Includes a brief discussion of exterior algebra and its application to differential geometry (p 189).

Wilson E B 1901

Vector Analysis
New York

The first formally published book entirely devoted to presenting the Gibbs-Heaviside system of vector analysis based on Gibbs' lectures and Heaviside's papers in the *Electrician* in 1893. (Wilson uses the term 'bivector', but by it means a vector with real and imaginary parts.)

Wilson E B and Lewis G N 1912

'The space-time manifold of relativity. The non-Euclidean geometry of mechanics and electromagnetics'
Proceedings of the American Academy of Arts and Sciences, **XLVIII**, pp 387-507.

This treatise uses a four-dimensional vector calculus developed by Lewis (see Lewis G N) by specializing the exterior calculus to four dimensions (with the scalar products of time-like vectors negative). This is a good example of the power of the exterior calculus.

Woo L and Freudenstein F 1969

Application of Line Geometry to Theoretical Kinematics and the Kinematic Analysis of Mechanical Systms
New York Scientific Center Technical Report 320-2982

This approach to mechanics uses line geometry. It does not use Grassmannian methods.

Zaddach A 1994

Grassmanns Algebra in der Geometrie
BI-Wissenschaftsverlag

In German only as of 2001.

Ziwet A 1885-6

'A Brief Account of H. Grassmann's Geometrical Theories'
Annals of Mathematics, **2**, (1885 pp 1-11; 1886 pp 25-34).

In the words of the author: "It is the object of the present paper to give in the simplest form possible, a succinct account of Grassmann's mathematical theories and methods in their application to plane geometry." (Follows in the main Schlegel's *System der Raumlehre*.)